

# New solution to quantum master equation for diffusion\*

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## Abstract

Based on the Kraus-form solution to the master equation describing diffusion we develop an integral-form solution by using the method of integration within ordered product of operators, i.e., the evolution law of density operator in diffusion channel can be considered as an integration transformation from an input to its output density operator. It brings much convenience for obtaining the time evolution law in the diffusion process via this formalism.

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## 1 Introduction

Due to the interaction between quantum system and environment, quantum decoherence or quantum diffusion are inevitable [1, 2]. Quantum decoherence phenomena (QDP), associated with quantum entanglement, teleportation, quantum noise, and fluctuations, is an important issue received much attention by scientists. Dynamics of this model can be described by the master equation or the associated Langevin or Fokker-Planck equations [2]. If these systems involve negligible correlations amongst their components, quantum memory (non-Markovian) effects can be ignored. In other words, Markovian approximation is valid. Considering a quantum harmonic system interacted with a thermal bath, its master equation describing time evolution of system's  $\rho$  in the diffusion channel is given by [4, 5]

$$\frac{d\rho}{dt} = -\kappa [a^\dagger a \rho - a^\dagger \rho a - a \rho a^\dagger + \rho a a^\dagger], \quad (1)$$

where  $a^\dagger$  is the Bose creation operator satisfying the relation  $[a, a^\dagger] = 1$ , and  $\kappa$  is a diffusion constant. In general, operator solutions to master equation are required to be in the infinite sum representation (named Kraus formal solution)

$$\rho(t) = \sum_{n=0}^{\infty} M_n \rho_0 M_n^\dagger, \quad (2)$$

where  $\rho_0$  is the initial density operator,  $M_n$  is the Kraus operator [6], and satisfies the normalized condition, i.e.,  $\sum_{n=0}^{\infty} M_n^\dagger M_n = 1$ . For different master equations, there correspond to different forms of Kraus operator solutions. However, the solutions in infinite sum representation to some master equations have still not been

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found. In Ref. [7] by using the entangled state representation and the method of integration within ordered product (IWOP) of operators, the solution to Eq. (1) is derived

$$\begin{aligned}\rho(t) &= \sum_{m,n=0}^{\infty} \sqrt{\frac{1}{m!n!} \frac{(\kappa t)^{m+n}}{(\kappa t + 1)^{m+n+1}}} a^{\dagger m} \left( \frac{1}{1 + \kappa t} \right)^{a^{\dagger} a} a^n \rho_0 a^{\dagger n} \left( \frac{1}{1 + \kappa t} \right)^{a^{\dagger} a} a^m \sqrt{\frac{1}{m!n!} \frac{(\kappa t)^{m+n}}{(\kappa t + 1)^{m+n+1}}} \\ &= \sum_{m,n=0}^{\infty} M_{m,n} \rho_0 M_{m,n}^{\dagger},\end{aligned}\quad (3)$$

where

$$M_{m,n} = \sqrt{\frac{1}{m!n!} \frac{(\kappa t)^{m+n}}{(\kappa t + 1)^{m+n+1}}} a^{\dagger m} \left( \frac{1}{1 + \kappa t} \right)^{a^{\dagger} a} a^n, \quad (4)$$

satisfying  $\sum_{m,n=0}^{\infty} M_{m,n}^{\dagger} M_{m,n} = 1$ , which is trace conservative. Even we have had the general form solution to the diffusion equation, it still remains difficulty when  $\rho_0$  is complicated, because performing sum for various Bose operators on the right hand side of Eq.(3) is tough, since  $a^{\dagger}$  is not commutative with  $a$ . To avoid this difficulty, in this paper, we will employ the IWOP method to develop the solution in Eq. (3) into a new integration transformation form. It brings much convenience for obtaining the time evolution law in the diffusion process via this new formalism and can help us to understand quantum diffusion more deeply.

## 2 Integration-form solution to the dissipative master equation (form 1)

In order to improve Eq.(3), we introduce coherent state representation [8]  $|\alpha\rangle = \exp\left[-\frac{|\alpha|^2}{2} + \alpha a^{\dagger}\right] |0\rangle$ , satisfying the eigenequation  $a|\alpha\rangle = \alpha|\alpha\rangle$ , whose completeness relation is

$$\int \frac{d^2\alpha}{\pi} |\alpha\rangle \langle\alpha| = 1. \quad (5)$$

Using the normally ordering form of vacuum projector  $|0\rangle \langle 0| =: e^{-a^{\dagger} a}:$  and the IWOP method, we can rewrite Eq.(5) as [9]

$$\int \frac{d^2\alpha}{\pi} |\alpha\rangle \langle\alpha| = \int \frac{d^2\alpha}{\pi} : e^{-|\alpha|^2 + a^{\dagger} \alpha + \alpha a^* - a^{\dagger} a} : = 1. \quad (6)$$

Noting that any operator can be expanded according to coherent states (so-called  $P$ -representation) [10], i.e.,

$$\rho_0 = \int \frac{d^2\alpha}{\pi} P(\alpha, 0) |\alpha\rangle \langle\alpha|, \quad (7)$$

using

$$\left( \frac{1}{1 + \kappa t} \right)^{a^{\dagger} a} |\alpha\rangle = e^{-a^{\dagger} a \ln(1 + \kappa t)} |\alpha\rangle = e^{-|\alpha|^2/2 + \alpha a^{\dagger} \frac{1}{1 + \kappa t}} |0\rangle, \quad (8)$$

and substituting Eq.(7) into Eq.(3), we have

$$\begin{aligned}\rho(t) &= \sum_{m,n=0}^{\infty} \frac{1}{m!n!} \frac{(\kappa t)^{m+n}}{(\kappa t + 1)^{m+n+1}} a^{\dagger m} \\ &\quad \times \left( \frac{1}{1 + \kappa t} \right)^{a^{\dagger} a} \int \frac{d^2\alpha}{\pi} P(\alpha, 0) |\alpha|^{2n} |\alpha\rangle \langle\alpha| \left( \frac{1}{1 + \kappa t} \right)^{a^{\dagger} a} a^m \\ &= \int \frac{d^2\alpha}{\pi} P(\alpha, 0) \sum_{m,n=0}^{\infty} \frac{|\alpha|^{2n}}{m!n!} \frac{(\kappa t)^{m+n}}{(\kappa t + 1)^{m+n+1}} a^{\dagger m} e^{-|\alpha|^2 + \alpha a^{\dagger} \frac{1}{1 + \kappa t}} |0\rangle \langle 0| e^{\alpha^* a \frac{1}{1 + \kappa t}} a^m.\end{aligned}\quad (9)$$

Then using  $|0\rangle\langle 0| =: e^{-a^\dagger a}$ : and noticing that  $a^\dagger$  is commutable with  $a$  within normal ordering symbol  $:$ , we can perform the summation in (9)

$$\begin{aligned}\rho(t) &= \int \frac{d^2\alpha}{\pi} e^{-|\alpha|^2} P(\alpha, 0) \sum_{m,n=0}^{\infty} \frac{|\alpha|^{2n}}{m!n!} \frac{(\kappa t)^{m+n}}{(\kappa t + 1)^{m+n+1}} : a^{\dagger m} a^m e^{(\alpha a^\dagger + \alpha^* a) \frac{1}{1+\kappa t} - a^\dagger a} : \\ &= \int \frac{d^2\alpha}{\pi} e^{-|\alpha|^2} P(\alpha, 0) \sum_{m=0}^{\infty} \frac{1}{m!} \frac{(\kappa t)^m}{(\kappa t + 1)^{m+1}} e^{\frac{\kappa t}{\kappa t + 1} |\alpha|^2} : a^{\dagger m} a^m e^{(\alpha a^\dagger + \alpha^* a) \frac{1}{1+\kappa t} - a^\dagger a} : \\ &= \frac{1}{\kappa t + 1} \int \frac{d^2\alpha}{\pi} e^{-\frac{|\alpha|^2}{\kappa t + 1}} P(\alpha, 0) : e^{\frac{\kappa t}{\kappa t + 1} a^\dagger a + \frac{1}{1+\kappa t} (\alpha a^\dagger + \alpha^* a) - a^\dagger a} : .\end{aligned}\quad (10)$$

Once the  $P$ -representation  $P(\alpha, 0)$  is known, we can derive  $\rho(t)$  by directly performing the integration in (10) by virtue of the IWOP method. For an initial coherent state  $\rho_0 = |z\rangle\langle z|$ , its  $P$ -representation is Delta function

$$P(\alpha, 0)_{|z\rangle\langle z|} = \pi \delta(\alpha^* - z^*) \delta(\alpha - z),$$

then from Eq. (10) we immediately have the final state in the diffusion channel

$$\begin{aligned}\rho(t)_{|z\rangle\langle z|} &= \frac{1}{\kappa t + 1} : e^{-|z|^2 \frac{1}{\kappa t + 1} + \frac{\kappa t}{\kappa t + 1} a^\dagger a + (za^\dagger + z^* a) \frac{1}{1+\kappa t} - a^\dagger a} : \\ &= \frac{1}{\kappa t + 1} e^{-|z|^2 \frac{1}{\kappa t + 1}} e^{za^\dagger \frac{1}{1+\kappa t}} e^{a^\dagger a \ln \frac{\kappa t}{\kappa t + 1}} e^{z^* a \frac{1}{1+\kappa t}},\end{aligned}\quad (11)$$

which is a mixed state involving a chaotic light characteristic of  $e^{a^\dagger a \ln \frac{\kappa t}{\kappa t + 1}}$ , thus we conclude that through a diffusion process a pure coherent state evolves into a mixed state, and we can confirm that Eq. (11) is qualified being a density operator as we can prove

$$\text{Tr} [\rho(t)_{|z\rangle\langle z|}] = \int \frac{d^2\alpha}{\pi} \langle \alpha | \rho(t)_{|z\rangle\langle z|} | \alpha \rangle = \frac{1}{\kappa t + 1} e^{-|z|^2 \frac{1}{\kappa t + 1}} \int \frac{d^2\alpha}{\pi} e^{-\frac{|\alpha|^2}{\kappa t + 1} + (z\alpha^* + z^*\alpha) \frac{1}{1+\kappa t}} = 1. \quad (12)$$

Then we can further prove the classical correspondence of  $\rho(t)_{|z\rangle\langle z|}$  really obey the classical diffusion equation, for this aim, we use the operator identity which can turn a density operator into its antinormal ordering form [13]

$$\rho = \int \frac{d^2\beta}{\pi} : \langle -\beta | \rho | \beta \rangle \exp[|\beta|^2 + \beta^* a - \beta a^\dagger + a^\dagger a] :, \quad (13)$$

where  $:$  denotes anti-normal ordering [11, 12], substituting (11) into the right-hand side of (13) we have

$$\begin{aligned}\rho_{|z\rangle\langle z|}(t) &= \frac{1}{1+\kappa t} \int \frac{d^2\beta}{\pi} : \exp[-\frac{\kappa t}{1+\kappa t} |\beta|^2 + \beta^* (a - \frac{z}{1+\kappa t}) + \beta (\frac{z^*}{1+\kappa t} - a^\dagger) - \frac{|z|^2}{1+\kappa t} + a^\dagger a] : \\ &= \frac{1}{\kappa t} : \exp[-\frac{1}{\kappa t} (z - a)(z^* - a^\dagger)] :, \end{aligned}\quad (14)$$

this is the anti-normally ordered form of  $\rho(t)_{|z\rangle\langle z|}$ . Its classical representation in the coherent state basis is

$$\rho(t)_{|z\rangle\langle z|} = \int \frac{d^2\alpha}{\pi} P(\alpha, t) |\alpha\rangle\langle\alpha|, \quad (15)$$

with

$$P(\alpha, t) = \frac{1}{\kappa t} \exp[-\frac{1}{\kappa t} (z - \alpha)(z^* - \alpha^*)]. \quad (16)$$

We can check that it really obeys the classical diffusion equation

$$\frac{\partial P(\alpha, t)}{\partial t} = \kappa \frac{\partial^2}{\partial \alpha \partial \alpha^*} P(\alpha, t). \quad (17)$$

For the origin of this equation we refer to the Appendix of this paper.

### 3 Integration-form solution to the dissipative master equation (form 2)

To go a step further, using the inverse relation of the  $P$ -representation in Eq. (7)

$$P(\alpha, 0) = e^{|\alpha|^2} \int \frac{d^2\beta}{\pi} \langle -\beta | \rho_0 | \beta \rangle e^{|\beta|^2 + \beta^* \alpha - \beta \alpha^*}, \quad (18)$$

where  $|\beta\rangle$  is also a coherent state ?? and substituting Eq. (14) into Eq. (10) yields

$$\begin{aligned} \rho(t) &= \frac{1}{\kappa t + 1} \int \frac{d^2\beta}{\pi} \langle -\beta | \rho_0 | \beta \rangle e^{|\beta|^2} \int \frac{d^2\alpha}{\pi} e^{\frac{\kappa t |\alpha|^2}{\kappa t + 1}} e^{\beta^* \alpha - \beta \alpha^*} : e^{\frac{1}{1+\kappa t}(\alpha a^\dagger + \alpha^* a) - \frac{1}{\kappa t + 1} a^\dagger a} : \\ &= -\frac{1}{\kappa t} \int \frac{d^2\beta}{\pi} \langle -\beta | \rho_0 | \beta \rangle e^{|\beta|^2} \\ &\quad \times : \exp \left[ \left( \frac{\kappa t + 1}{-\kappa t} \right) \left( \frac{a^\dagger}{1 + \kappa t} + \beta^* \right) \left( \frac{a}{1 + \kappa t} - \beta \right) - \frac{1}{\kappa t + 1} a^\dagger a \right] : \\ &= \frac{-1}{\kappa t} \int \frac{d^2\beta}{\pi} \langle -\beta | \rho_0 | \beta \rangle e^{|\beta|^2} : \exp \left\{ \frac{1}{\kappa t} [|\beta|^2 (\kappa t + 1) + \beta a^\dagger - \beta^* a - a^\dagger a] \right\} : , \end{aligned} \quad (19)$$

this is the integration-form solution to the diffusion master equation, connecting input state  $\rho_0$  with its output state  $\rho(t)$ .

### 4 Applications

Eq. (15) brings much convenience for obtaining the time evolution law in the diffusion process. As an example, we consider the case of an initial number state  $|l\rangle \langle l|$  undergoing through the diffusion channel, here  $|l\rangle = a^{\dagger l} |0\rangle / \sqrt{l!}$ . Due to

$$\langle l | \beta \rangle = e^{-|\beta|^2/2} \frac{\beta^l}{\sqrt{l!}} \quad (20)$$

and

$$\langle -\beta | l \rangle \langle l | \beta \rangle = \frac{(-1)^l |\beta|^{2l}}{l!} e^{-|\beta|^2}. \quad (21)$$

Substituting (16) into (14) and using the formula

$$\int \frac{d^2\beta}{\pi} \beta^n \beta^{*m} \exp(\zeta |\beta|^2 + \xi \beta + \eta \beta^*) = e^{\frac{-\xi \eta}{\zeta}} \sum_{k=0}^{\min[n,m]} \frac{n! m! \xi^{m-k} \eta^{n-k}}{k! (n-k)! (m-k)! (-\zeta)^{m+n-k+1}}. \quad (22)$$

Using the definition of the two-variable Hermite polynomials

$$H_{m,n}(x, y) = \sum_{l=0}^{\min(m,n)} \frac{m! n! (-1)^l}{l! (m-l)! (n-l)!} x^{m-l} y^{n-l}, \quad (23)$$

as well as the definition of Laguerre polynomials

$$L_l(x) = \sum_{k=0}^l \binom{l}{k} \frac{(-x)^k}{k!} \quad (24)$$

and

$$L_l(xy) = \frac{(-1)^l}{l!} H_{l,l}(x, y), \quad (25)$$

we can derive

$$\begin{aligned} \rho(t) &= \frac{-1}{\kappa t} \int \frac{d^2\beta}{\pi} \langle -\beta | \beta \rangle \langle l | \beta \rangle e^{|\beta|^2} : \exp \left\{ \frac{1}{\kappa t} [|\beta|^2 (\kappa t + 1) + \beta a^\dagger - \beta^* a - a^\dagger a] \right\} : \\ &= \frac{(-1)^{l+1}}{l! \kappa t} \int \frac{d^2\beta}{\pi} |\beta|^{2l} : \exp \left\{ \frac{1}{\kappa t} [|\beta|^2 (\kappa t + 1) + \beta a^\dagger - \beta^* a - a^\dagger a] \right\} : \\ &= \frac{(\kappa t)^l}{(\kappa t + 1)^{l+1}} : L_l \left( \frac{-a^\dagger a}{\kappa t (\kappa t + 1)} \right) e^{\frac{-1}{\kappa t + 1} a^\dagger a} : , \end{aligned} \quad (26)$$

which is named Laguerre-polynomial-weighted chaotic state, since  $e^{\frac{-1}{\kappa t+1}a^\dagger a}$  represents a chaotic photon field, here the symbol " : : " denotes normal ordering symbol. Thus we see  $|l\rangle\langle l|$  evolves into the mixed state (29), so this diffusion process manifestly embodies quantum decoherence. We can further calculate the average photon number  $Tr[\rho(t)a^\dagger a] = l + \kappa t$ , which tells that in the diffusion process, the photon number  $l \rightarrow l + \kappa t$ .

As the second example, If the initial state is a squeezed state with a squeezing parameter  $\lambda$ , whose density operator is given by

$$\rho_0 = \text{sech}\lambda e^{\frac{1}{2}a^{\dagger 2} \tanh \lambda} |0\rangle\langle 0| e^{\frac{1}{2}a^2 \tanh \lambda}. \quad (27)$$

Substituting (22) into (14) and using

$$\langle -\beta | \rho_0 | \beta \rangle = \text{sech}\lambda e^{\frac{1}{2}(\beta^{*2} + \beta^2) \tanh \lambda - |\beta|^2}, \quad (28)$$

as well as the integration formula

$$\int \frac{d^2 z}{\pi} \exp(\zeta |z|^2 + \xi z + \eta z^* + f z^2 + g z^{*2}) = \frac{1}{\sqrt{\zeta^2 - 4fg}} \exp\left[\frac{-\zeta\xi\eta + f\eta^2 + g\xi^2}{\zeta^2 - 4fg}\right], \quad (29)$$

we can finally obtain a mixed state

$$\begin{aligned} \rho(t) &= -\frac{\text{sech}\lambda}{\kappa t} \int \frac{d^2 \beta}{\pi} : \exp\left\{\frac{\kappa t + 1}{\kappa t} |\beta|^2 + \frac{\beta a^\dagger - \beta^* a}{\kappa t} + \frac{\tanh \lambda}{2} (\beta^{*2} + \beta^2) - \frac{a^\dagger a}{\kappa t}\right\} : \\ &= -\text{sech}\lambda \sqrt{\frac{1}{G}} : \exp\left[\frac{\frac{\kappa t + 1}{\kappa t} a^\dagger a + \frac{\tanh \lambda}{2} (a^{\dagger 2} + a^2)}{G} - \frac{a^\dagger a}{\kappa t}\right] : \\ &= -\text{sech}\lambda \sqrt{\frac{1}{G}} e^{\frac{\tanh \lambda}{2G} a^{\dagger 2}} : \exp\left[\frac{(\kappa t + 1) a^\dagger a}{G \kappa t} - \frac{a^\dagger a}{\kappa t}\right] : e^{\frac{\tanh \lambda}{2G} a^2}, \end{aligned} \quad (30)$$

which is a mixed state, where

$$G \equiv (\kappa t + 1)^2 - (\kappa t)^2 \tanh^2 \lambda.$$

Actually, Eq.(24) can be considered as a squeezed thermal state.

In summary, for master equation describing the dissipative channel, we developed the form solution of Kraus into a new integration form by using the IWOP technique. It will be more concise to derive various density operator  $\rho(t)$  and can help us to understand deeply the quantum decoherence and reduce calculation vastly.

## 5 Appendix

We now demonstrate that the classical correspondence to the diffusion master equation

$$\frac{d\rho}{dt} = -\kappa(a^\dagger a \rho - a^\dagger \rho a - a \rho a^\dagger + \rho a a^\dagger) \quad (31)$$

is Eq. (17). In fact, using

$$\rho(t) = \int \frac{d^2 \alpha}{\pi} P(\alpha, t) |\alpha\rangle\langle \alpha|, \quad (32)$$

Eq. (31) becomes to

$$\frac{d\rho}{dt} = -\kappa \int \frac{d^2 \alpha}{\pi} P(\alpha, t) (a^\dagger a |\alpha\rangle\langle \alpha| - a^\dagger |\alpha\rangle\langle \alpha| a - a |\alpha\rangle\langle \alpha| a^\dagger + |\alpha\rangle\langle \alpha| a a^\dagger). \quad (33)$$

Considering

$$\begin{aligned} a^\dagger |\alpha\rangle\langle \alpha| &= : a^\dagger : e^{-|\alpha|^2 + \alpha a^\dagger + \alpha^* a - a^\dagger a} : = (\alpha^* + \frac{\partial}{\partial \alpha}) |\alpha\rangle\langle \alpha|, \\ |\alpha\rangle\langle \alpha| a &= (\alpha + \frac{\partial}{\partial \alpha^*}) |\alpha\rangle\langle \alpha|, \end{aligned} \quad (34)$$

we can see

$$\begin{aligned}
& a^\dagger a |\alpha\rangle\langle\alpha| - a^\dagger |\alpha\rangle\langle\alpha| a - a |\alpha\rangle\langle\alpha| a^\dagger + |\alpha\rangle\langle\alpha| a a^\dagger \\
&= \alpha a^\dagger |\alpha\rangle\langle\alpha| - (\alpha^* + \frac{\partial}{\partial\alpha}) |\alpha\rangle\langle\alpha| a - |\alpha|^2 |\alpha\rangle\langle\alpha| + (\alpha + \frac{\partial}{\partial\alpha^*}) |\alpha\rangle\langle\alpha| a^\dagger \\
&= \alpha(\alpha^* + \frac{\partial}{\partial\alpha}) |\alpha\rangle\langle\alpha| - (\alpha^* + \frac{\partial}{\partial\alpha})(\alpha + \frac{\partial}{\partial\alpha^*}) |\alpha\rangle\langle\alpha| - |\alpha|^2 |\alpha\rangle\langle\alpha| + (\alpha + \frac{\partial}{\partial\alpha^*})(\alpha^* |\alpha\rangle\langle\alpha|) \\
&= -\frac{\partial^2}{\partial\alpha\partial\alpha^*} |\alpha\rangle\langle\alpha|.
\end{aligned} \tag{35}$$

Inserting (35) into (33), we obtain

$$\frac{d\rho}{dt} = \kappa \int \frac{d^2\alpha}{\pi} \frac{\partial^2 P(\alpha, t)}{\partial\alpha\partial\alpha^*} |\alpha\rangle\langle\alpha|. \tag{36}$$

On the other hand, we have

$$\frac{d\rho}{dt} = \int \frac{d^2\alpha}{\pi} \frac{\partial P(\alpha, t)}{\partial t} |\alpha\rangle\langle\alpha|. \tag{37}$$

Comparing (36) with (37) we derive Eq. (17), the classical diffusion equation.

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